Some algebraic identities in anomalous gauge theories

1989 J. Phys. A: Math. Gen. 22 L371

(http://iopscience.iop.org/0305-4470/22/9/004)

J. Phys. A: Math. Gen. **22** (1989) L371-L375. Printed in the UK

LETTER TO THE EDITOR

Some algebraic identities in anomalous gauge theories†

Yao-Zhong Zhang

Center of Theoretical Physics, CCAST(World Laboratory), Beijing, People's Republic of China, and Institute of Modern Physics, Northwest University, Xian, People's Republic of China

Received 26 January 1989

Abstract. Some useful algebraic relations between the anomaly of $D_\mu J^\mu$ and anomalous commutators are derived using only the canonical commutators and the equation of motion. Their applications are discussed.

The consistent anomaly in chiral gauge theories has attracted much interest in the past couple of years (see [1] for reviews and references to the original literature). On the other hand, Faddeev [2] conjectures that there may exist an anomalous term in the commutator of the Gauss law constraints. This suggestion has encouraged others [3, 4] to compute this commutator explicitly. In [3], Jo perturbatively computes the various kinds of commutators; for example, the commutator between the current and the electric field and the commutator between two electric fields. In this letter I shall establish some algebraic relations between these commutators and the anomaly of $D_\mu J^\mu$ (where $J_a^\mu = \bar{\psi}\gamma^\mu(1-\gamma_s)/2t^a\psi$). The derivation is purely algebraic and involves no regularisation at any point. The typical uses to which these relations can be put are to obtain the anomaly of $D_\mu J^\mu$ from the E-E and J_0-E commutators and to compute the Schwinger-Jackiw-Johnson (SJJ) term in the commutator of the constraints from the anomaly of $D_\mu J^\mu$.

In the Weyl gauge, the equations of motion of gauge fields interacting with sources are

$$\partial_0 E_i^a(x) = J_i^a(x) + (D_k F^{ki}(x))^a \qquad E_i^a(x) = -\partial_0 A_i^a(x). \tag{1}$$

It is easily shown that the Gauss law constraint, $G^a(x) = (D_i E^i(x))^a + J_0^a(x)$, satisfies the anomalous Ward identity [5]

$$\partial_0 G^a(x) = H_{\text{con}}^a(x) \tag{2}$$

where

$$H_{\text{con}}^a(x) = (D_\mu J^\mu(x))^a = \partial_0 J_0^a(x) + f^{abc} A_i^b(x) J_i^c(x) \tag{3}$$

is the consistent anomaly in the Weyl gauge.

The occurrence of the anomaly in (2) means that the constraint is not a conserved quantity. When we impose on the physical state the Gauss law condition at an initial time, this condition is not satisfied at a later time. Therefore, we cannot consistently quantise the anomalous gauge theory in the conventional way [6].

† Supported in part by the Chinese National Postdoctoral Foundation.

The usual canonical commutators are

$$[A_i^a(x), A_j^b(y)] = 0 \tag{4a}$$

$$[E_i^a(x), A_j^b(y)] = \delta_{ij}\delta^{ab}\delta^3(x-y). \tag{4b}$$

Starting with the above equations (1)-(4), we can derive the following formal relations.

Relation I. $[E_i^a(x), E_j^b(y)] = [J_i^a(x), A_j^b(y)]$.

Proof. Differentiating (4*b*) with respect to time, and replacing the time derivatives of $E_i^a(x)$ and $A_i^a(x)$ by (1), we obtain relation I at once.

Relation II. $(\partial/\partial E_j^b(x))H_{\text{con}}^a(x)\delta^3(x-y) = D_i^{ac}(x)[E_i^c(x), E_j^b(y)] + [J_0^a(x), E_j^b(y)]$.

Proof. By (1)

$$[J_0^a(x), E_j^b(y)] = [J_0^a(x), -\partial_0 A_j^b(y)] = -\partial_0[J_0^a(k), A_j^b(y)] + [\partial_0 J_0^a(x), A_j^b(y)].$$

The $J_0^a(x)$ operate only on the fermion fields and thus commute with $A_i^b(y)$. Using (3) and relation I, we have

$$\begin{aligned}[\partial_0 J_0^a(x), A_j^b(y)] &= [H_{\text{con}}^a(x) - \partial_i J_i^a(x) - f^{adc}A_i^d(x)J_i^c(x), A_j^b(y)]\\ &= [H_{\text{con}}^a(x), A_j^b(y)] - D_i^{ac}(x)[E_i^c(x), E_j^b(y)].\end{aligned}$$

On the other hand,

$$[H_{\text{con}}^a(x), A_j^b(y)] = \frac{\partial}{\partial E_i^c(x)} H_{\text{con}}^a(x)[E_i^c(x), A_j^b(y)] = \frac{\partial}{\partial E_j^b(x)} H_{\text{con}}^a(x)\delta^3(x-y).$$

Thus relation II is seen to hold.

Relation III.

$$\begin{aligned}\partial_0 W^{ab}(x,y) = {}& \frac{\partial}{\partial A_i^c(y)} H_{\text{con}}^b(y) D_i^{ac}(x)\delta^3(x-y) - \frac{\partial}{\partial A_i^c(x)} H_{\text{con}}^a(x) D_i^{bc}(y)\delta^3(y-x)\\ & + f^{adc}\frac{\partial}{\partial E_i^c(y)} H_{\text{con}}^b(y) E_i^d(x)\delta^3(x-y)\\ & - f^{bdc}\frac{\partial}{\partial E_i^c(x)} H_{\text{con}}^a(x) E_i^d(y)\delta^3(x-y)\\ & - f^{abc}H_{\text{con}}^c(x)\delta^3(x-y)\end{aligned}$$

where $W^{ab}(x, y)$ is the SJJ term.

Proof. Let $[G^a(x), G^b(y)] = f^{abc}G^c(x)\delta^3(x-y) + W^{ab}(x,y)$. Differentiating this equation with respect to time, and replacing $\partial_0 G$ by (2), we obtain

$$\partial_0 W^{ab}(x,y) = [G^a(x), H_{\text{con}}^b(y)] - [G^b(y), H_{\text{con}}^a(x)] - f^{abc}H_{\text{con}}^c(x)\delta^3(x-y).$$

Obviously,

$$[G^a(x), H_{\text{con}}^b(y)] = [G^a(x), A_i^c(y)]\frac{\partial}{\partial A_i^c(y)} H_{\text{con}}^b(y) + [G_{(x)}^a, E_i^c(y)]\frac{\partial}{\partial E_i^c(y)} H_{\text{con}}^b(y).$$

By (1), (2) and the canonical commutators,

$$[G^a(x), A_i^c(y)] = D_i^{ac}(x)\delta^3(x-y)$$

$$\begin{aligned}[G^a(x), E_i^c(y)] &= [G^a(x), -\partial_0 A_i^c(y)] = -\partial_0[G^a(x), A_i^c(y)] + [\partial_0 G^a(x), A_i^c(y)] \\ &= -\partial_0 D_i^{ac}(x)\delta^3(x-y) + [H^a_{\text{con}}(x), A_i^c(y)] \\ &= f^{adc}E_i^d(x)\delta^3(x-y) + \frac{\partial}{\partial E_i^c(x)} H^a_{\text{con}}(x)\delta^3(x-y)\end{aligned}$$

we obtain

$$\begin{aligned}[G^a(x), H^b_{\text{con}}(y)] &= \frac{\partial}{\partial A_i^c(y)} H^b_{\text{con}}(y) D_i^{ac}(x)\delta^3(x-y) + f^{adc}\frac{\partial}{\partial E_i^c(y)} H^b_{\text{con}}(y) E_i^d(x)\delta^3(x-y) \\ &\quad + \frac{\partial}{\partial E_i^c(x)} H^a_{\text{con}}(x)\frac{\partial}{\partial E_i^c(y)} H^b_{\text{con}}(y)\delta^3(x-y).\end{aligned}$$

In the same way, we have

$$\begin{aligned}[G^b(y), H^a_{\text{con}}(x)] &= \frac{\partial}{\partial A_i^c(x)} H^a_{\text{con}}(x) D_i^{bc}(y)\delta^3(y-x) + f^{bdc}\frac{\partial}{\partial E_i^c(x)} H^a_{\text{con}}(x) E_i^d(y)\delta^3(x-y) \\ &\quad + \frac{\partial}{\partial E_i^c(y)} H^b_{\text{con}}(y)\frac{\partial}{\partial E_i^c(x)} H^a_{\text{con}}(x)\delta^3(x-y).\end{aligned}$$

Combining all these equations gives relation III.

Relation II reveals the connection between the divergence anomaly and the E-E and J_0-E commutators, and suggests an initimate connection between the anomaly and the SJJ term in the commutator of the Gauss law constraints. These relations are established by purely algebraic manipulations, without involving regularisiation at any point. This means that while the calculation of any anomalous object must involve some kind of regularisation at some stage, two or three related anomalies can be connected algebraically. As the first application to these relations, we shall obtain the anomaly of $D_\mu J^\mu$ from the E-E and J_0-E commutators. In [3], Jo derives the following E-E and J_0-E commutators using the BJL (Bjorken-Johnson-Low) limit method:

$$[E_i^a(x), E_j^b(y)] = \frac{1}{24\pi^2}\varepsilon^{ijk}\,\text{Tr}\{t^a, t^b\}A_k\delta^3(x-y) \tag{5}$$

$$\begin{aligned}[E_i^a(x), J_0^b(y)] &= \frac{-1}{24\pi^2}\varepsilon^{ijk}\,\text{Tr}\{t^a, t^b\}A_j\partial_k\delta^3(x-y) \\ &\quad - \frac{1}{12\pi^2}\varepsilon^{ijk}\,\text{Tr}\{t^a, t^b\}\partial_j A_k\delta^3(x-y) \\ &\quad - \frac{1}{16\pi^2}\varepsilon^{ijk}(\text{Tr}\{t^a, t^b\}A_jA_k - \text{Tr}\, t^bA_jt^aA_k)\delta^3(x-y).\end{aligned} \tag{6}$$

Substituting (5) and (6) into the left-hand side of relation II gives

$$\frac{\partial}{\partial E_j^b(x)} H^a_{\rm con}(x)\delta^3(x-y)$$

$$= \frac{1}{24\pi^2}\varepsilon^{jki}\,\mathrm{Tr}\{t^a, t^b\}\partial_k A_i \delta^3(x-y)$$

$$+\frac{1}{48\pi^2}\varepsilon^{jki}(\mathrm{Tr}\{t^a, t^b\}A_k A_i + \mathrm{Tr}\ t^a A_i t^b A_k)\delta^3(x-y) \qquad (7)$$

which yields

$$H^a_{\rm con}(x) = \frac{1}{24\pi^2}\varepsilon^{ijk}\,\mathrm{Tr}\ E_i(t^a\partial_j A_k + \partial_j A_k t^a + \tfrac{1}{2}A_j A_k t^a + \tfrac{1}{2}A_j t^a A_k + \tfrac{1}{2}t^a A_j A_k) \qquad (8)$$

which is the well known expression of the divergence anomaly in the Weyl gauge. Therefore, we computed the anomaly of $D_\mu J^\mu$ using the anomalous E-E and J_0-E commutators. This may be regarded as a perturbative derivation of the divergence anomaly, since the anomalous E-E and J_0-E commutators used here were calculated using the BJL method. Of course, this is not the whole story. We wanted to show the relations between the anomalous E-E and J_0-E commutators and the anomaly.

The second application to the relations I–III is to obtain the SJJ term in the commutator of the Gauss law constraints from a known expression for the divergence anomaly. Relation III provides such a possibility. As for the divergence anomaly, we shall use the well known expression (8). Given the divergence anomaly, to simplify the left-hand side of relation III is only a trivial task. After a long but direct calculation, we find

$$W^{ab}(x, y) = \frac{1}{48\pi^2}\varepsilon^{ijk}[\mathrm{Tr}[t^a, t^b](\partial_i A_j A_k + A_i\partial_j A_k + A_i A_j A_k)$$

$$+ \mathrm{Tr}\ t^a\partial_i(A_j t^b A_k)]\delta^3(x-y) \qquad (9)$$

which is just the expression found by Jo through the BJL method and differs with the expression originally predicted [2] only by some trivial terms. This may be regarded as a non-perturbative derivation of the W^{ab}.

In conclusion, we have obtained some algebraic relations between the divergence anomaly and the anomalous commutators. These relations provide new ways of computing the divergence anomaly and the SJJ term in the G-G commutator, and show clearly the intimate connection between the anomaly and the anomalous commutators.

In a recent paper, [7] we showed the connection between the anomalous Jacobian and the divergence anomaly by obtaining the former from the latter. The analysis there plus the one here demonstrate that the anomalous commutators and Jacobian are the other faces of the divergence anomaly.

References

[1] Jackiw R 1986 *Preprint* CTP/1436 (*CECS Scientific Series* ed C Teitelboim (New York: Plenum) to be published)
Treiman S, Jackiw R, Zumino B and Witten E 1985 *Current Algebra and Anomalies* (Princeton, NJ: Princeton University Press/Singapore: World Scientific)

[2] Faddeev L D 1984 *Phys. Lett.* **145B** 81
[3] Jo S G 1985 *Phys. Lett.* **163B** 353; 1985 *Nucl. Phys.* B **259** 616
[4] Kobayashi M, Seo K and Sugamoto A 1986 *Nucl. Phys.* B **73** 607
[5] Hwang D S 1987 *Nucl. Phys.* B **286** 231
Mitra P 1988 *Phys. Rev. Lett.* **60** 265
[6] Jackiw R and Rajaraman R 1985 *Phys. Rev. Lett.* **54** 1219
Faddeev L D and Shatashvili S L 1986 *Phys. Lett.* **167B** 225
[7] Zhang Y-Z 1988 *Preprint* Northwest University NWU-IMP-88-21